On weakly symmetric Riemannian manifolds

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Abstract. In this paper its proved three theorems about weakly symmetric manifolds. The first one is a sufficiency condition for a $(WS)_n$ to be a $G(PS)_n$ and a $(PS)_n$. The second one is about the Ricci tensor of a conformally flat $(WS)_n$ with non zero scalar curvature, and the last one is about $(WS)_n$ with cyclic Ricci tensor.

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Introduction

A *pseudo symmetric manifold* which was introduced in [3] is a non-flat Riemannian manifold V_n $(n > 2)$ in which the curvature tensor R_{hijk} satisfies the condition

$$R_{hijk,l} = 2A_l R_{hijk} + A_h R_{lijk} + A_i R_{hljk} + A_j R_{hilk} + A_k R_{hijl},$$

where A is a non-zero 1-form and ',' denotes covariant differentiation with respect to the metric tensor of the manifold and A is called it's associated 1-form. The n-dimensional manifolds of this kind are denoted by $(PS)_n$.

A *Generalized pseudo symmetric manifold* was which introduced in [1] is a non-flat Riemannian manifold V_n $(n > 2)$ in which the curvature tensor R_{hijk} satisfies the condition

$$R_{hijk,l} = 2A_l R_{hijk} + B_h R_{lijk} + C_i R_{hljk} + D_j R_{hilk} + A_k R_{hijl},$$

where A, B, C and D are 1-forms (non-zero simultaneously). The n-dimensional manifolds of this kind are denoted by $G(PS)_n$. Its shown in [2], the defining condition of a $G(PS)_n$ can be expressed in the following form

$$R_{hijk,l} = 2A_l R_{hijk} + B_h R_{lijk} + B_i R_{hljk} + A_j R_{hilk} + A_k R_{hijl},$$

where A and B are 1-forms (non-zero simultaneously) and are called the associated 1-forms of the manifold.

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The notion of weakly symmetric manifold was introduced in [7]. A non-flat Riemannian manifold V_n $(n > 2)$ is called a *weakly symmetric manifold* if the curvature tensor R_{hijk} satisfies the condition

$$R_{hijk,l} = A_l R_{hijk} + B_h R_{lijk} + C_i R_{hljk} + D_j R_{hilk} + E_k R_{hijl},$$

where A, B, C, D and E are 1-forms (non-zero simultaneously) and are called the associated 1-forms of the manifold. The n-dimensional manifolds of this kind are denoted by $(WS)_n$. Its shown in [4] and [5] that, the defining condition of a $(WS)_n$ can be expressed in the following way

$$R_{hijk,l} = A_l R_{hijk} + B_h R_{lijk} + B_i R_{hljk} + D_j R_{hilk} + D_k R_{hijl}. \tag{1.1}$$

Although the definition of a $(WS)_n$ is similar to that of a generalized pseudo-symmetric manifold, but the defining condition of a $(WS)_n$ is weaker than that of a generalized pseudo-symmetric manifold. In the case of $B = D = \frac{1}{2}A$, a $(WS)_n$ is just a pseudo-symmetric manifold, so the notion of a $(WS)_n$ is a natural generalization of that of a $(PS)_n$.

In the present paper some results on weakly symmetric Riemannian manifolds are established. In section 2 there is a sufficient condition for a $(WS)_n$ to be a $G(PS)_n$ and a $(PS)_n$. In section 3 its proved that the Ricci tensor of a conformally flat $(WS)_n$ with non-zero scalar curvature, has a special form. Finally in section 4, its shown that there does not exist any weakly symmetric manifold with cyclic Ricci tensor, if the manifold's Ricci curvature is non zero.

2 Ricci-associate of associated 1-forms in $(WS)_n$

Let

$$V_i = R_{hi} A^h. \tag{2.1}$$

Then the 1-form with coefficients V_i is called the *Ricci-associate of the 1-form* with coefficients A_i.

Let A, B and D be the associated 1-forms of a $(WS)_n$. We call their Ricci-associate respectively by U, V and W. Thus we have

$$U_i = R_{hi} A^h, \qquad V_i = R_{hi} B^h, \qquad W_i = R_{hi} D^h, \tag{2.2}$$

where U_i, V_i and W_i are the components of the 1-forms U, V and W.

Theorem 2.1 *In a $(WS)_n$ with non-zero scalar curvature, if $\frac{1}{2}U = W$, then this manifold will be a $G(PS)_n$ and if $\frac{1}{2}U = W = V$, then it will be a $(PS)_n$.*

Proof Transvecting (1.1) with g^{hk}, we have

$$R_{ij,l} = A_l R_{ij} + B_i R_{jl} + D_j R_{li} + B^h R_{lijh} + D^h R_{hijl}. \tag{2.3}$$

From (2.3) we get

$$R_{,l} = A_l R + 2(B^h + D^h) R_{hl}. \tag{2.4}$$

Multiplying (1.1) by $g^{hk}g^{il}$, we find

$$\frac{R_{,l}}{2} = D_l R + (A^h + B^h - D^h)R_{hl}. \tag{2.5}$$

In addition, contracting (1.1) with $g^{hk}g^{jl}$, we have

$$\frac{R_{,l}}{2} = B_l R + (A^h + D^h - B^h)R_{hl}. \tag{2.6}$$

From (2.4), (2.5) and (2.6) we obtain

$$R(A_l - 2B_l) = 2(A^h - 2B^h)R_{hl}. \tag{2.7}$$

And

$$R(A_l - 2D_l) = 2(A^h - 2D^h)R_{hl}. \tag{2.8}$$

If $\frac{1}{2}U = W$, then from (2.2) and (2.8) we deduce that $\frac{1}{2}A_l = D_l$ for each l, and thus the manifold is a $G(PS)_n$, and if $\frac{1}{2}U = W = V$, then by(2.2), (2.7) and (2.8) its clear that the manifold is a $(PS)_n$. □

3 Conformally flat $(WS)_n$

In this section we suppose that a weakly symmetric Riemannian manifold is conformally flat.

Its known ([8], p. 41) that in a conformally flat (M^n, g) $(n \geq 3)$

$$R_{ij,k} - R_{ki,j} = \frac{1}{2(n-1)}[g_{ij}R_{,k} - g_{ki}R_{,j}]. \tag{3.1}$$

On the other hand, with the help of (2.3) and the Ricci identity we have

$$\begin{aligned} R_{ij,k} - R_{ki,j} &= (A_k - D_k)R_{ij} + (D_j - A_j)R_{ki} \\ &\quad + B^h R_{hijk} + 2D^h R_{hijk}. \end{aligned} \tag{3.2}$$

Multiplying both side of (2.4) and (3.2) by B^j, we express (3.1) as follows

$$\begin{aligned} B^j R_{ki}(A_j - D_j) = &-\frac{1}{2(n-1)}[B_i(A_k R + 2(B^h + D^h)R_{hk}) \\ &- g_{ki}B^j(A_j R + 2(B^h + D^h)R_{hj})] + B^h R_{hi}(A_k - D_k) \\ &+ B^h B^j R_{hijk} + 2D^h B^j R_{hijk}. \end{aligned} \tag{3.3}$$

In a conformally flat (M^n, g) $(n \geq 3)$ we have ([6], p. 92)

$$\begin{aligned} R_{kijh} = &\frac{1}{n-2}[R_{ij}g_{kh} - R_{jk}g_{hi} + R_{kh}g_{ij} - R_{hi}g_{jk}] \\ &+ \frac{R}{(n-1)(n-2)}[g_{jk}g_{hi} - g_{ij}g_{kh}]. \end{aligned} \tag{3.4}$$

If the scalar curvature R is non-zero, by using equation (3.4), we can rewrite (3.3) as follows

$$R_{ik}\,\{\,B^j(A_j - D_j) + \frac{1}{n-2}\,B^j(B_j + 2D_j)\,\}$$

$$= \frac{-R}{2(n-1)}\,B_iA_k + \frac{1}{n-2}\,B^hR_{hi}[\,(n-2)A_k + B_k \,-\,(n-4)\,D_k\,]$$

$$+\frac{1}{(n-1)(n-2)}\,B_i(B^h + n\,D^h)\,R_{hk}$$

$$+\frac{R}{2(n-1)(n-2)}\,[\,(n-2)B^jA_j + 2\,B^jB_j + 4\,B^jD_j\,]\,g_{ik}$$

$$-\frac{1}{(n-1)(n-2)}\,[(B^h + D^h)B^kR_{hk}\,]\,g_{ik} - \frac{R}{(n-1)(n-2)}\,B_i\,[\,B_k + D_k\,],$$

or

$$R_{ij} = a_1\,g_{ij} \,+\, b_1\,B_iA_j \,+\, b_2\,B_i\,B_j \,+\, b_3\,B_iD_j \,+\, b_4\,B_i\,B^hR_{hj}$$

$$+\,b_5\,B_iD^hR_{hj} + b_6\,B^hR_{hi}A_j + b_7\,B^hR_{hi}\,B_j + b_8\,B^hR_{hi}\,D_j, \tag{3.5}$$

where A, B and D are associated 1-forms of the manifold and $a_1,\ b_1,\ \cdots,\ b_8$ are smooth functions on manifold in terms of R, A_jB^j, B_jB^j and D_jB^j.

Hence we can state the following theorem:

Theorem 3.1 *In a conformally flat $(WS)_n$ of non-zero scalar curvature whith associated 1-forms A, B and D, the Ricci tensor S with coefficients R_{ij} has the form* (3.5).

Now, if the scalar curvature R is non-zero constant, then the formula (3.5) reduces to the following form

$$R_{ij} = a_1\,g_{ij} \,+\, b_1\,B_i\,B_j \,+\, b_2\,B_iD_j \,+\, b_3\,B_i\,B^hR_{hj} \,+\, b_4\,B_iD^hR_{hj}$$

$$+\,b_5\,B^hR_{hi}A_j \,+\, b_6\,B^hR_{hi}\,B_j \,+\, b_7\,B^hR_{hi}\,D_j,$$

where $a_1,\ b_1,\ \cdots,\ b_7$ are smooth functions on manifold in terms of R, A_jB^j, B_jB^j and D_jB^j.

4 Cyclic Ricci tensor on $(WS)_n$

A Riemannian manifold is said to be *cyclic Ricci tensor* if the condition

$$R_{ij,k} + R_{jk,i} + R_{ki,j} = 0, \tag{4.1}$$

holds for it's Ricci tensor. Transvecting (4.1) with g^{ij} we get $R_{,k} = 0$, which means that the scalar curvature in a manifold with cyclic Ricci tensor, is constant.

Theorem 4.1 *There dose not exist any $(WS)_n$ with cyclic Ricci tensor, if its Ricci curvature is non zero.*

Proof Using (2.3) and Bianchi Identity, we get

$$A_i^* R_{jk} + A_j^* R_{ki} + A_k^* R_{ij} = 0, \tag{4.2}$$

where $A_k^* = A_k + B_k + D_k$. Since the scalar curvature of manifold is constant, by the equations (2.4), (2.5) and (2.6) we have

$$A_l^* R + 2A^{h*} R_{hl} = 0, \tag{4.3}$$

where $A^{h*} = A^h + B^h + D^h$. Transvecting (4.2) with A^{k*} and by the aid of the equation (4.3), we can easily obtain

$$A^{k*} A_k^* R_{ij} = A_i^* A_j^* R. \tag{4.4}$$

In this case, multiplying (4.4) with A^{i*} and using the equation (4.3), we get

$$\frac{3}{2}\, A^{i*} A_i^* A_j^* R = 0. \tag{4.5}$$

Since the metric of the manifold is positive definite, then $A^{i*} A_i^* \neq 0$. From (4.5) we have

$$A_j^* R = 0. \tag{4.6}$$

Therefore from the equation (4.4), $R_{ij} = 0$, which is a contradiction. □

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